RESTRICTED FLOW IN A NON LINEAR CAPACITATED TRANSPORTATION PROBLEM WITH BOUNDS ON RIM CONDITIONS

KAVITA GUPTA*

S.R. ARORA**

Abstract:

This paper discusses restricted flow in a fixed charge capacitated transportation problem with bounds on total source availabilities and total destination requirements . The objective function is the sum of two linear fractional functions consisting of variable costs and fixed charges respectively. Sometimes, situations arise when one wishes to keep reserve stocks at the sources for emergencies , thereby restricting the total transportation flow to a known specified level. A related transportation problem is formulated and it is shown that to each basic feasible solution called corner feasible solution to related transportation problem , there is a corresponding feasible solution to this restricted flow problem . The optimal solution to restricted flow problem may be obtained from the optimal solution to related transportation problem. An algorithm is presented to solve non linear capacitated transportation problem with restricted flow . Numerical illustration is included in support of theory.

Keywords: Capacitated transportation problem, restricted f low, fixed charge, related problem , corner feasible solution .

* Department of Mathematics, Jagan Institute of Management Studies, 3 Institutional Area, Sector-5, Rohini , Delhi, India.

** Ex-Principal, Hans Raj College, University of Delhi, Delhi-110007, India.

1 Introduction:

There is a wide scope of capacitated transportation problem with bounds on rim conditions. It can be used extensively in tele-communication networks, production – distribution systems, rail and urban road system when there is a limited capacity of resources such as vehicles , docks, equipment capacity etc. These are bounded variable transportation problems . Many researchers like Dahiya and Verma [3], Misra and Das [8] have contributed in this field .

The fixed charge (for example, a set up cost) arises for a transportation problem in which each origin has a fixed charge coefficient in addition to the usual cost coefficient . Fixed charge transportation problems have been studied by Arora et .al [1-2], Sandrock [9] and many others.

Another class of transportation problems is a non linear programming problem where the objective function to be optimized is a ratio of two linear functions . Optimization of a ratio of criteria often describes some kind of an efficiency measure for a system . Non linear programs finds its application in a variety of real world problems such as stock cutting problem , resource allocation problems , routing problem for ships and planes , cargo – loading problem , inventory problem and many other problems. Dahiya and Verma [4] studied paradox in non linear capacitated transportation problem. Arora et .al [7] studied indefinite quadratic transportation problems.

Many researchers like Arora [5,6] , Thirwani [10] have studied restricted flow problems. Sometimes , situations arise when reserve stocks are to be kept at sources for emergencies . This gives rise to restricted flow problem where the total flow is restricted to a known specified level. This motivated us to develop an algorithm to solve a non linear capacitated transportation problem with restricted flow.

2 Problem Formulation:

Consider a non linear capacitated transportation problem given by

$$(P1):\ \min z = \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}} + \frac{\sum_{i\in I} F_i}{\sum_{i\in I} G_i} \right]$$

subject to

$$a_i \le \sum_{i\in I} x_{ij} \le A_i\ ;\forall\, i \in I \tag{1}$$

$$b_j \le \sum_{j\in J} x_{ij} \le B_j\ ;\forall\, j \in J \tag{2}$$

$$l_{ij} \le x_{ij} \le u_{ij} \text{ and integers } \forall\, i \in I,\, j \in J \tag{3}$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P\left(< \min\left(\sum_{i\in I} A_{i,} \sum_{j\in J} B_j \right)\right) \tag{4}$$

where I = {1, 2, … m} is the index set of m origins.

J = {1, 2, …, n} is the index set of n destinations.

x_{ij} = number of units transported from origin i to the destination j.

c_{ij} = per unit pilferage cost when shipment is sent from i^{th} origin to the j^{th} destination.

d_{ij} = the variable profit per unit amount transported from i^{th} origin to the j^{th} destination.

F_i = the capital investment

G_i = return on investment

l_{ij} and u_{ij} are the bounds on number of units to be transported from i^{th} origin to j^{th} destination.

a_i and A_i are the bounds on the availability at the i^{th} origin, i □ I

b_j and B_j are the bounds on the demand at the j^{th} destination, j □ J

For the formulation of F_i (i=1,2 … m), we assume that F_i (i = 1, 2 .. m) has p numberof steps so that

$$F_i = \sum_{l=1}^{p} F_{il}\ \delta_{il}, \quad i = 1, 2 \ldots m$$

where, $\delta_{il} = \begin{cases} 1 & \text{if } F_i = \sum_{j=1}^{n} x_{ij} > a_{il} \\ 0 & \text{otherwise} \end{cases}$ for l=1,2,3…….p, i=1,2,……m

Here, $0 = a_{i1} < a_{i2} \ldots < a_{ip}$

$a_{i1}, a_{i2} \ldots, a_{ip}$ $(i = 1,2, \ldots m)$ are constants and F_{il} are the fixed costs $\forall$ i= 1, 2 …m, l = 1,2 ..p

Sometimes , situations arise when one wishes to keep reserve stocks at the origins for emergencies, there by restricting the total transportation flow to a known specified level, say P $\left(< \min\left(\sum_{i\in I} A_i, \sum_{j\in J} B_j \right) \right)$.This flow constraint changes the structure of the transportation problem.

It is assumed that $\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij} > 0$ for every feasible solution X satisfying (1),(2),(3) and (4) and all upper bounds u_{ij} ; $(i,j) \in I\times J$ are finite.

In order to solve problem (P1) , we convert it in to a related problem (P2) given by

$$\text{(P2): } \min\left[\frac{\sum_{i\in I'}\sum_{j\in J'} c'_{ij}y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d'_{ij}y_{ij}} + \frac{\sum_{i\in I'} F'_i}{\sum_{i\in I'} G'_i} \right]$$

subject to

$$\sum_{j\in J'} y_{ij} = A'_i ; \forall i \in I'$$

$$\sum_{i\in I'} y_{ij} = B'_j ; \forall j \in J'$$

$$l_{ij} \le y_{ij} \le u_{ij} ; \forall\ i, j\ \in I\times J$$

$$0 \le y_{m+1,\, j} \le B_j - b_j ; \forall j \in J$$

$$0 \le y_{i,\, n+1} \le A_i - a_i ; \forall i \in I$$

$$y_{m+1,\, n+1} = 0$$

$A_i' = A_i \ \forall i\in I, \quad A'_{m+1} = \sum_{j\in J} B_j - P \ , B'_j = B_j \ \forall j\in J \ , B'_{n+1} = \sum_{i\in I} A_i - P$

$c'_{ij} = c_{ij} \ , \forall i\in I, j\in J, \ c'_{m+1,j} = c'_{i,n+1} = 0 \quad \forall i\in I, \ \forall j\in J, \ c'_{m+1,n+1} = M$

$d'_{ij} = d_{ij} \ \forall i\in I, j\in J, \quad d'_{m+1,j} = d'_{i,n+1} = 0 \quad \forall i\in I, \ \forall j\in J; \ d'_{m+1,n+1} = M$

$F_i' = F_i \ \forall i\in I, \ F'_{m+1} = 0; \ G_i' = G_i; \forall i \in I, G'_{m+1} = 0$

where $I' = \{1, 2, \ldots m, m+1\}$, $J' = \{1, 2, \ldots n, n+1\}$

3 Theoretical Development:

Definition: Corner feasible solution : A basic feasible solution $\{y_{ij}\}$ $i\in I', j \in J'$ to (P2) is called a corner feasible solution (cfs) if $y_{m+1,n+1} = 0$

Theorem 1. A non corner feasible solution of (P2) cannot provide a basic feasible solution to (P1).

Proof: Let $\{y_{ij}\}_{I' \times J'}$ be a non corner feasible solution to (P2). Then $y_{m+1,n+1} = \lambda$ (>0)

Thus $\sum_{i\in I'} y_{i,n+1} = \sum_{i\in I} y_{i,n+1} + y_{m+1,n+1}$

$$= \sum_{i\in I} y_{i,n+1} + \lambda$$

$$= \sum_{i\in I} A_i - P$$

Therefore, $$\sum_{i\in I} y_{i,n+1} = \sum_{i\in I} A_i - (P+\lambda) \qquad (5)$$

Now, for $i \in I$,

$\sum_{j\in J'} y_{ij} = A_i' = A_i$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J'} y_{ij} = \sum_{i\in I} A_i \qquad (6)$$

(5) and (6) implies that $\sum_{i\in I}\sum_{j\in J} y_{ij} = P + \lambda$

This implies that total quantity transported from the sources in I to the destination in J is P + $\lambda >$ P, a contradiction to assumption that total flow is P and hence $\{y_{ij}\}_{I' \times J'}$ can not provide a feasible solution to (P1).

Lemma 1: There is a one –to-one correspondence between the feasible solution to (P1) and the corner feasible solution to (P2).

Proof: Let $\{x_{ij}\}_{I \times J}$ be a feasible solution of (P1).So $\{x_{ij}\}_{I \times J}$ will satisfy (1) to (4).

Define $\{y_{ij}\}_{I' \times J'}$ by the following transformation

$$y_{ij} = x_{ij} \quad, i \in I,\ j \in J$$

$$y_{i,n+1} = A_i - \sum_{j \in J} x_{ij} \quad , \ i \in I$$

$$y_{m+1,j} = B_j - \sum_{i \in I} x_{ij} \quad , \ j \in J$$

$$y_{m+1,n+1} = 0$$

It can be shown that $\{y_{ij}\}_{I' \times J'}$ so defined is cfs to (P2).

Relation (1) to (4) implies that

$$l_{ij} \le y_{ij} \le u_{ij} \qquad \text{for all } i \in I,\ j \in J$$

$$0 \le y_{i,n+1} \le A_i - a_i \quad , i \in I$$

$$0 \le y_{m+1,j} \le B_j - b_j \quad , j \in J$$

$$y_{m+1,n+1} \ge 0$$

Also for $i \in I$

$$\sum_{j \in J'} y_{ij} = \sum_{j \in J} y_{ij} + y_{i,n+1} = \sum_{j \in J} x_{ij} + A_i - \sum_{j \in J} x_{ij} = A_i = A_i'$$

For i = m+1

$$\sum_{j \in J'} y_{m+1,j} = \sum_{j \in J} y_{ij} + y_{m+1,n+1} = \sum_{j \in J} (B_j - \sum_{i \in I} x_{ij})$$

$$= \sum_{j \in J} B_j - \sum_{i \in I} \sum_{j \in J} x_{ij}$$

$$= \sum_{j \in J} B_j - P$$

$$= A'_{m+1}$$

$\Rightarrow \sum_{j\in J'} y_{ij} = A_i^{'} \ ; \ \forall i \in I'$

Similarly, it can be shown that $\sum_{i\in I'} y_{ij} = B_j^{'} \ ; \ \forall j \in J'$

Therefore, $\{y_{ij}\}_{I'xJ'}$ is a cfs to (P2).

Conversely, let $\{y_{ij}\}_{I'xJ'}$ be a cfs to (P2).Define x_{ij} , $i\in I$, $j \in J$ by the following transformation.

$x_{ij}= y_{ij}$, $i\in I$, $j \in J$

It implies that $l_{ij} \le x_{ij} \le u_{ij}$, $i\in I$, $j \in J$

Now for $i\in I$, the source constraints in (P2´) gives

$$\sum_{j\in J'} y_{ij} = A_i^{'} = A_i$$

$$\sum_{j\in J} y_{ij} + y_{i,\,n+1} = A_i$$

$$\Rightarrow a_i \le \sum_{j\in J} y_{ij} \le A_i \qquad (\text{since } 0 \le y_{i,n+1} \le A_i - a_i \ , \ i\in I)$$

Hence, $a_i \le \sum_{j\in J} x_{ij} \le A_i \ , \ i\in I$

Similarly, for $j \in J$, $b_j \le \sum_{i\in I} x_{ij} \le B_j$

For i= m+1,

$$\sum_{j\in J'} y_{m+1,\,j} = A_{m+1}^{'} = \sum_{j\in J} B_j - P$$

$$\Rightarrow \sum_{j\in J} y_{m+1,\,j} = \sum_{j\in J} B_j - P \qquad (\text{because } y_{m+1,n+1} = 0)$$

Now, for $j \in J$ the destination constraints in (P2´) gives

$$\sum_{i\in I} y_{ij} + y_{m+1,\,j} = B_j$$

Therefore, $\sum_{i\in I}\sum_{j\in J} y_{ij} + \sum_{j\in J} y_{m+1,\,j} = \sum_{j\in J} B_j$

$$\sum_{i\in I}\sum_{j\in J} y_{ij} = \sum_{j\in J} B_j - \sum_{j\in J} y_{m+1,\,j} = P$$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} x_{ij} = P$$

Therefore $\{x_{ij}\}_{I\,xJ}$ is a feasible solution to (P2)

Remark 1: If (P2) has a cfs ,then since $c'_{m+1,n+1}$=M and $d'_{m+1,n+1}$= M, it follows that non corner feasible solution can not be an optimal solution of (P2) .

Lemma 2: The value of the objective function of (P1) at a feasible solution $\{x_{ij}\}_{I\,x\,J}$ is equal to the value of the objective function of (P2) at its corresponding cfs $\{y_{ij}\}_{I'xJ'}$ and conversely.

Proof: The value of the objective function of (P2) at a feasible solution $\{y_{ij}\}_{I'xJ'}$ is

$$z = \left[\frac{\sum_{i\in I'}\sum_{j\in J'} c'_{ij}y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d'_{ij}y_{ij}} + \frac{\sum_{i\in I'} F'_i}{\sum_{i\in I'} G'_i}\right]$$

$$= \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}} + \frac{\sum_{i\in I} F_i}{\sum_{i\in I} G_i}\right] \text{ because } \left\{\begin{array}{l} c'_{ij} = c_{ij}, \forall i \in I, j \in J \\ d'_{ij} = d_{ij}, \forall i \in I, j \in J \\ x_{ij} = y_{ij}, \forall i \in I, j \in J \\ c'_{i,n+1} = c'_{m+1,j} = 0; \forall i \in I, j \in J \\ d'_{i,n+1} = d'_{m+1,j} = 0; \forall i \in I, j \in J \\ y_{m+1,n+1} = 0 \\ F'_{m+1} = 0, F'_i = F_i, \forall i \in I \\ G'_{m+1} = 0, G'_i = G_i, \forall i \in I \end{array}\right\}$$

= the value of the objective function of (P1) at the corresponding cfs $\{x_{ij}\}_{I\,xJ}$

The converse can be proved in a similar way.

Lemma 3: There is a one –to-one correspondence between the optimal solution to (P1) and optimal among the corner feasible solution to (P2).

Proof: Let $\{\overset{\circ}{x}_{ij}\}_{I\times J}$ be an optimal solution to (P1) yielding objective function value z^0 and $\{\overset{\circ}{y}_{ij}\}_{I'\times J'}$ be the corresponding cfs to (P2). Then by Lemma 2, the value yielded by $\{\overset{\circ}{y}_{ij}\}_{I'\times J'}$ is z^0 .If possible, let $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ be not an optimal solution to (P2). Therefore, there exists a cfs $\{y'_{ij}\}$, say ,to (P2) with the value $z^1 < z^0$. Let $\{x'_{ij}\}$ be the corresponding feasible solution to (P1).Then by lemma 2,

$z^1 = \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x'_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x'_{ij}} + \frac{\sum_{i\in I} F_i}{\sum_{i\in I} G_i}\right]$, a contradiction to the assumption that $\{\overset{\circ}{x}_{ij}\}_{I\times J}$ is an optimal solution of (P1).Similarly, an optimal corner feasible solution to (P2) will give an optimal solution to (P1).

Theorem 2: Optimizing (P2) is equivalent to optimizing (P1) provided (P1) has a feasible solution.

Proof: As (P1) has a feasible solution , by lemma 1,there exists a cfs to (P2).Thus by remark 1,an optimal solution to (P2) will be a cfs. Hence , by lemma 3,an optimal solution to (P1) can be obtained.

Theorem 3: A feasible solution $X^0 = \{\overset{\circ}{x}_{ij}\}_{I\times J}$ of problem (P2) with objective function value $\frac{N^\circ}{D^\circ} + \frac{F^\circ}{G^\circ}$ will be a local optimum basic feasible solution iff the following conditions holds.

$$\delta^1_{ij} = \frac{\theta_{ij}\left[D^\circ(c_{ij} - z^1_{ij}) - N^\circ(d_{ij} - z^2_{ij})\right]}{D^\circ\left[D^\circ + \theta_{ij}(d_{ij} - z^2_{ij})\right]} + \frac{G^\circ \Delta F_{ij} - F^\circ \Delta G_{ij}}{G^\circ(G^\circ + \Delta G_{ij})} \geq 0; \forall (i,j) \in N_1$$

$$\delta^2_{ij} = -\frac{\theta_{ij}\left[D^\circ(c_{ij} - z^1_{ij}) - N^\circ(d_{ij} - z^2_{ij})\right]}{D^\circ\left[D^\circ - \theta_{ij}(d_{ij} - z^2_{ij})\right]} + \frac{G^\circ \Delta F_{ij} - F^\circ \Delta G_{ij}}{G^\circ(G^\circ + \Delta G_{ij})} \geq 0; \forall (i,j) \in N_2$$

and if X^0 is an optimal solution of (P2),then $\delta^1_{ij} \geq 0; \forall (i,j) \in N_1$ and $\delta^2_{ij} \geq 0; \forall (i,j) \in N_2$ where $N^\circ = \sum_{i\in I}\sum_{j\in J} c_{ij}x^\circ_{ij}$, $D^\circ = \sum_{i\in I}\sum_{j\in J} d_{ij}x^\circ_{ij}$, $F^\circ = \sum_{i\in I} F_i$, $G^\circ = \sum_{i\in I} G_i$,B denotes the set of cells (i,j) which are basic and N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively.

$u_i^1, u_i^2, v_j^1, v_j^2; i \in I, j \in J$ are the dual variables such that $u_i^1 + v_j^1 = c_{ij}$, $\forall (i,j) \in B$; $u_i^2 + v_j^2 = d_{ij}$, $\forall (i,j) \in B$; $u_i^1 + v_j^1 = z_{ij}^1$, $\forall (i,j) \notin B$; $u_i^2 + v_j^2 = z_{ij}^2$, $\forall (i,j) \notin B$; $\Delta F_{ij}, \Delta G_{ij}$ are the corresponding changes in $\sum_{i \in I} F_i$ and $\sum_{i \in I} G_i$ when some non basic variable x_{ij} undergoes change by an amount of θ_{ij} .

Proof: Let $X^0 = \{\overset{\circ}{x}_{ij}\}_{I \times J}$ be a basic feasible solution of problem (P2) with equality constraints. Let z^0 be the corresponding value of objective function. Then

$$z^\circ = \left[\frac{\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}^\circ}{\sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}^\circ} + \frac{\sum_{i \in I} F_i}{\sum_{i \in I} G_i} \right] = \frac{N^\circ}{D^\circ} + \frac{F^\circ}{G^\circ} \text{ (say)}$$

$$= \left[\frac{\sum_{i \in I} \sum_{j \in J} (c_{ij} - u_i^1 - v_j^1) x_{ij}^\circ + \sum_{i \in I} \sum_{j \in J} (u_i^1 + v_j^1) x_{ij}^\circ}{\sum_{i \in I} \sum_{j \in J} (d_{ij} - u_i^1 - v_j^1) x_{ij}^\circ + \sum_{i \in I} \sum_{j \in J} (u_i^2 + v_j^2) x_{ij}^\circ} + \frac{\sum_{i \in I} F_i}{\sum_{i \in I} G_i} \right]$$

$$= \left[\frac{\sum \sum_{(i,j) \in N_1} (c_{ij} - u_i^1 - v_j^1) l_{ij} + \sum \sum_{(i,j) \in N_2} (c_{ij} - u_i^1 - v_j^1) u_{ij} + \sum_{i \in I} \sum_{j \in J} (u_i^1 + v_j^1) x_{ij}^\circ}{\sum \sum_{(i,j) \in N_1} (d_{ij} - u_i^2 - v_j^2) l_{ij} + \sum \sum_{(i,j) \in N_2} (d_{ij} - u_i^2 - v_j^2) u_{ij} + \sum_{i \in I} \sum_{j \in J} (u_i^2 + v_j^2) x_{ij}^\circ} + \frac{\sum_{i \in I} F_i}{\sum_{i \in I} G_i} \right]$$

$$= \left[\frac{\sum \sum_{(i,j) \in N_1} (c_{ij} - z_{ij}^1) l_{ij} + \sum \sum_{(i,j) \in N_2} (c_{ij} - z_{ij}^1) u_{ij} + \sum_{i \in I} a_i u_i^1 + \sum_{j \in J} b_j v_j^1}{\sum \sum_{(i,j) \in N_1} (d_{ij} - z_{ij}^2) l_{ij} + \sum \sum_{(i,j) \in N_2} (d_{ij} - z_{ij}^2) u_{ij} + \sum_{i \in I} a_i u_i^2 + \sum_{j \in J} b_j v_j^2} + \frac{\sum_{i \in I} F_i}{\sum_{i \in I} G_i} \right]$$

Let some non basic variable $x_{ij} \in N_1$ undergoes change by an amount θ_{rs} where θ_{rs} is given by

$\min\{u_{rs} - l_{rs}; x_{ij}^\circ - l_{ij}$ for all basic cells (i, j) with a $(-\theta)$ entry in $\theta -$ loop;

$u_{ij} - x_{ij}^\circ$ for all basic cells (i, j) with a $(+\theta)$ entry in $\theta -$ loop$\}$

Let $\Delta F_{rs}, \Delta G_{rs}$ be the corresponding changes in $\sum_{i\in I} F_i$ and $\sum_{i\in I} G_i$ .Then new value of the objective function $\hat{z}$ will be given by

$$\hat{z} = \frac{N^\circ + \theta_{rs}(c_{rs} - z^1_{rs})}{D^\circ + \theta_{rs}(d_{rs} - z^2_{rs})} + \frac{F^\circ + \Delta F_{rs}}{G^\circ + \Delta G_{rs}}$$

$$\hat{z} - z^\circ = \left[\frac{N^\circ + \theta_{rs}(c_{rs} - z^1_{rs})}{D^\circ + \theta_{rs}(d_{rs} - z^2_{rs})} - \frac{N^\circ}{D^\circ}\right] + \left[\frac{F^\circ + \Delta F_{rs}}{G^\circ + \Delta G_{rs}} - \frac{F^\circ}{G^\circ}\right]$$

$$= \frac{\theta_{rs}\left[D^\circ(c_{rs} - z^1_{rs}) - N^\circ(d_{rs} - z^2_{rs})\right]}{D^\circ\left[D^\circ + \theta_{rs}(d_{rs} - z^2_{rs})\right]} + \frac{G^\circ \Delta F_{rs} - F^\circ \Delta G_{rs}}{G^\circ(G^\circ + \Delta G_{rs})} = \delta^1_{rs}(\text{say})$$

Similarly, when some non basic variable $x_{pq} \in N_2$ undergoes change by an amount θ_{pq} then

$$\hat{z} - z^\circ = -\frac{\theta_{pq}\left[D^\circ(c_{pq} - z^1_{pq}) - N^\circ(d_{pq} - z^2_{pq})\right]}{D^\circ\left[D^\circ - \theta_{pq}(d_{pq} - z^2_{pq})\right]} + \frac{G^\circ \Delta F_{pq} - F^\circ \Delta G_{pq}}{G^\circ(G^\circ + \Delta G_{pq})} = \delta^2_{pq}(\text{say})$$

Hence X^0 will be local optimal solution iff $\delta^1_{ij} \geq 0; \forall (i,j) \in N_1$ and $\delta^2_{ij} \geq 0; \forall (i,j) \in N_2$.If X^0 is global optimal solution of (P2), then it is locally optimal and hence the result follows.

4 Algorithm:

Step 1: Given a non linear capacitated transportation problem (P1), form a related transportation problem (P2). Find a basic feasible solution of (P2) with respect to variable cost only. Let B be the corresponding basis.

Step 2: Find the corresponding fixed cost. Let it be denoted by F^0/G^0 where $F^\circ = \sum_{i\in I} F_i$; $G^\circ = \sum_{i\in I} G_i$

Step 3(a): Calculate θ_{ij}, $u^1_i, u^2_i, v^1_j, v^2_j, z^1_{ij}, z^2_{ij}; i \in I, j \in J$ such that

$u_i^1 + v_j^1 = c_{ij}, \forall (i,j) \in B;$

$u_i^2 + v_j^2 = d_{ij}, \forall (i,j) \in B$

$u_i^1 + v_j^1 = z_{ij}^1, \forall (i,j) \notin B$

$u_i^2 + v_j^2 = z_{ij}^2, \forall (i,j) \notin B$

Step3(b) Calculate $N^\circ, D^\circ, F^\circ, G^\circ$ where $N^\circ = \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}, D^\circ = \sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}, F^\circ = \sum_{i\in I} F_i, G^\circ = \sum_{i\in I} G_i$

Step3(c): Calculate A_{ij}^1 and A_{ij}^2 where $A_{ij}^1 = \theta_{ij}(c_{ij} - z_{ij}^1); \forall (i,j) \notin B$ and $A_{ij}^2 = \theta_{ij}(d_{ij} - z_{ij}^2); \forall (i,j) \notin B$.

A_{ij}^1 is the change in numerator variable cost that occurs when a non basic cell (i,j) undergoes a change equal to θ_{ij} .Similarly , A_{ij}^2 is the change in denominator variable cost that occurs when a non basic cell (i,j) undergoes a change equal to θ_{ij}.

Step 4: Find $\Delta F_{ij} = F(NB) - F^\circ$ where F(NB) is the total capital investment obtained when some non basic cell (i,j) undergoes a change equal to θ_{ij}. Also find $\Delta G_{ij} = G(NB) - G^\circ$ where G(NB) is the total return on investment when some non basic cell (i,j) undergoes a change.

Step 5(a): Find $\Delta_{ij} = D^\circ(c_{ij} - z_{ij}^1) - N^\circ(d_{ij} - z_{ij}^2); \forall (i,j) \notin B$.

Step 5(b): Calculate δ_{ij}^1 and δ_{ij}^2 such that

$$\delta_{ij}^1 = \left[\frac{\theta_{ij}\Delta_{ij}}{D^\circ\left[D^\circ + A_{ij}^2\right]} + \frac{G^\circ \Delta F_{ij} - F^\circ \Delta G_{ij}}{G^\circ(G^\circ + \Delta G_{ij})}\right]; \forall (i,j) \in N_1 \text{ and}$$

$$\delta_{ij}^2 = \left[-\frac{\theta_{ij}\Delta_{ij}}{D^\circ\left[D^\circ - A_{ij}^2\right]} + \frac{G^\circ \Delta F_{ij} - F^\circ \Delta G_{ij}}{G^\circ(G^\circ + \Delta G_{ij})}\right]; \forall (i,j) \in N_2$$

where N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively.

If $\delta_{ij}^{1} \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^{2} \geq 0; \forall (i,j) \in N_2$ then current solution is the optimal solution to (P2) and subsequently to (P1).Then go to step (6). Otherwise some $(i,j) \in N_1$ for which $\delta_{ij}^{1} < 0$ or some $(i,j) \in N_2$ for which $\delta_{ij}^{2} < 0$ will undergo change .Go to step 2.

Step 6: Find the optimal cost z of (P1) yielded by the basic feasible solution .

5 Numerical illustration:

Consider the problem (P1) for m=3, n=3.Table 1 gives the values of c_{ij} , d_{ij} (i=1,2,3;j=1,2,3)

Table 1: values of c_{ij} , d_{ij}, Ai ,B_j

	D_1	D_2	D_3	A_i
O_1	5 4	9 2	9 1	30
O_2	4 3	6 7	2 4	40
O_3	2 2	1 9	1 4	50
B_j	30	20	30	

Note: values in the upper left corners are c_{ij} 's and values in lower left corners are d_{ij} 's for i=1,2,3.and j=1,2,3.

Also, $3 \leq \sum_{j=1}^{3} x_{1j} \leq 30$, $10 \leq \sum_{j=1}^{3} x_{2j} \leq 40$, $10 \leq \sum_{j=1}^{3} x_{3j} \leq 50$, $5 \leq \sum_{i=1}^{3} x_{i1} \leq 30$, $5 \leq \sum_{i=1}^{3} x_{i2} \leq 20$, $5 \leq \sum_{i=1}^{3} x_{i3} \leq 30$

$1 \leq x_{11} \leq 10$, $2 \leq x_{12} \leq 10$, $0 \leq x_{13} \leq 5$,$0 \leq x_{21} \leq 15$, $3 \leq x_{22} \leq 15$, $1 \leq x_{23} \leq 20$,$0 \leq x_{31} \leq 20$, $0 \leq x_{32} \leq 13$, $0 \leq x_{33} \leq 25$

$F_i = \sum_{l=1}^{3} F_{il}\delta_{il}$ and $G_i = \sum_{l=1}^{3} G_{il}\delta_{il}$ for i= 1,2,3 where

$$\delta_{i1} = \begin{cases} 1 & \text{if } \sum_{j=1}^{3} x_{ij} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_{i2} = \begin{cases} 1 & \text{if } \sum_{j=1}^{3} x_{ij} > 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_{i3} = \begin{cases} 1 & \text{if } \sum_{j=1}^{3} x_{ij} > 20 \\ 0 & \text{otherwise} \end{cases}$$

$F_{11}= 100$, $F_{12} = 50$, $F_{13} = 50$, $F_{21} = 150$, $F_{22} = 100$, $F_{23} = 50$,$F_{31}= 200$, $F_{32}= 150$, $F_{33} = 100$

$G_{11}= 100$, $G_{12} = 100$, $G_{13} =100$, $G_{21} = 200$, $G_{22} = 150$, $G_{23} = 100$,$G_{31}= 200$, $G_{32}= 200$, $G_{33} = 150$

Let the restricted flow be P = 40 where P = 40 < min $\left(\sum_{i=1}^{3} A_i = 120, \sum_{j=1}^{3} B_j = 80 \right)$

Introduce a dummy origin and a dummy destination in Table 1 with $c_{i4} = 0 = d_{i4}$ for all i = 1,2 ,3 and $c_{4j} = 0 = d_{4j}$ for all j = 1,2,3 . $c_{44} = d_{44} = M$ where M is a large positive number. Also we have $0 \le x_{14} \le 27$, $0 \le x_{24} \le 30$, $0 \le x_{34} \le 40$, $0 \le x_{41} \le 25$, $0 \le x_{42} \le 15$, $0 \le x_{43} \le 25$ and $F_{4j} = 0$,$G_{4j} = 0$ for j=1,2,3,4. Also, $B_4 = \sum_{i=1}^{3} A_i - P$ =120-40 = 80 and $A_4 = \sum_{j=1}^{3} B_j - P$ = 80-40 = 40. In this way , we form the problem (P2).

Find a basic feasible solution to problem (P2) which is given in table 2 below.

Table 2: A basic feasible solution of problem (P2)

	D_1	D_2	D_3	D_4	u_i^1	u_i^2	F^0	G^0
O_1	5 4 $\underline{1}$	9 2 $\underline{2}$	9 1 $\underline{0}$	0 0 **27**	0	0	100	100
O_2	4 3 $\underline{0}$	6 7 $\underline{3}$	2 4 **17**	0 0 **20**	0	0	250	350
O_3	2 2 **4**	1 9 $\overline{13}$	1 4 $\underline{0}$	0 0 **33**	0	0	350	400
O_4	0 0 $\overline{25}$	0 0 **2**	0 0 **13**	M M	-2	-4	0	0
v_j^1	2	2	2	0				
v_j^2	2	4	4	0				

Note: entries of the form $\underline{a}$ and $\overline{b}$ represent non basic cells which are at their lower and upper bounds respectively. Entries in bold are basic cells.

$N^0 = 96$, $D^0 = 222$, $F^0 = 700$, $G^0 = 850$, $z = 1.25596$

Table 3: optimality condition

NB	O_1D_1	O_1D_2	O_1D_3	O_2D_1	O_2D_2	O_3D_2	O_3D_3	O_4D_1
θ_{ij}	4	2	5	4	2	3	10	10
$c_{ij} - z_{ij}^1$	3	7	7	2	4	-1	-1	0
$d_{ij} - z_{ij}^2$	2	-2	-3	1	3	5	0	2
A_{ij}^1	12	14	35	8	8	-3	-10	0
A_{ij}^2	8	-4	-15	4	6	15	0	20
ΔF_{ij}	0	0	0	50	0	50	0	0

ΔG_{ij}	0	0	0	100	0	100	0	0
Δ_{ij}	474	1746	1842	348	600	-702	-222	-192
δ^1_{ij}	0.0371	0.0722	0.2004	-0.006	0.0237		-0.045	
δ^2_{ij}						0.04583		0.0428

Enter the cell O_3D_3 and find the improved solution which is given in table 4 below.

Table 4: Optimal solution of problem (P2)

	D_1	D_2	D_3	D_4	u^1_i	u^2_i	F^0	G^0
O_1	5 $\underline{1}$ 4	9 $\underline{2}$ 2	9 $\underline{0}$ 1	0 **27** 0	0	0	100	100
O_2	4 $\underline{0}$ 3	6 $\underline{3}$ 7	2 **7** 4	0 $\overline{30}$ 0	1	0	150	200
O_3	2 **4** 2	1 $\overline{13}$ 9	1 **10** 4	0 **23** 0	0	0	450	550
O_4	0 $\overline{25}$ 0	0 **2** 0	0 **13** 0	M M	-1	-4	0	0
v^1_j	2	1	1	0				
v^2_j	2	4	4	0				

$N^0 = 86$, $D^0 = 222$, $F^0 = 700$, $G^0 = 850$, $z = 1.2108$

Table 5: optimality condition

NB	O_1D_1	O_1D_2	O_1D_3	O_2D_1	O_2D_2	O_2D_4	O_3D_2	O_4D_1
θ_{ij}	4	2	5	4	2	10	13	10
$c_{ij} - z^1_{ij}$	3	8	8	1	4	-1	0	-1

$d_{ij} - z_{ij}^2$	2	-2	-3	1	3	0	5	2
A_{ij}^1	12	16	40	4	8	-10	0	-10
A_{ij}^2	8	-4	-15	4	6	0	65	20
ΔF_{ij}	0	0	0	0	0	0	0	0
ΔG_{ij}	0	0	0	0	0	0	0	0
Δ_{ij}	494	1948	2034	136	630	-222	-430	-394
δ_{ij}^1	0.0387	0.0805	0.2213	0.0108	0.02489			
δ_{ij}^2						0.04504	0.16038	0.08786

Since $\delta_{ij}^1 \geq 0; \forall (i, j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall (i, j) \in N_2$, the solution given in table 4 is an optimal solution.

6 Conclusion:

In order to solve a non linear capacitated transportation problem with restricted flow , a related transportation problem is formed and it is shown that the given problem and related transportation problem are equivalent. The algorithm developed in this paper minimizes the variable cost and maximizes the variable profit earned while shipping goods from various sources to different destinations . Simultaneously , it maximizes the rate of return on capital investment of fixed nature.

References:

- Arora , S.R., Ahuja, A.,A paradox in a fixed charge transportation problem , Indian Journal of Pure and Applied Mathematics, 31(7) (2000) 809-822
- Arora, S.R., Khurana ,A ., Three dimensional fixed charge bi-criterion indefinite quadratic transportation problem , Yugoslav Journal of Operations Research , 14 (1) (2004) 83-97

- Dahiya , K.,Verma ,V.,Capacitated transportation problem with bounds on rim conditions, Europeon Journal of Operational Research , 178(2007)718-317
- Dahiya, K., Verma ,V., Paradox in a non linear capacitated transportation problem, Yugoslav Journal of Operations Research , 169(2)(2006)189-210
- Khurana , A., Arora , S.R., The sum of a linear and linear fractional transportation problem with restricted and enhanced flow, Journal of Interdisciplinary Mathematics, 9(2006) 373-383
- Khurana , A., Thirwani , D. and Arora , S.R., An algorithm for solving fixed charge bi – criterion indefinite quadratic transportation problem with restricted flow, International Journal of Optimization : Theory , Methods and Applications,1(2009)367-380
- Khurana , A., Arora , S.R., Fixed charge bi – criterion indefinite quadratic transportation problem with enhanced flow, Revista Investigacion Operacional , 32(2011)133-145
- Misra , S., Das , C., Solid transportation problem with lower and upper bounds on rim conditions – a note , New Zealand Operational Research 9(2)(1981)137-140
- Sandrock , K., A simple algorithm for solving small fixed charge transportation problem , Journal of Operational Research Society, 39,467-475
- Thirwani , D. and Arora , S.R., and Khanna , S., An algorithm for solving fixed charge bi – criterion transportation problem with restricted flow , Optimization , 40, (1997)193-206